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Bäcklund transformations for the KP and mKP hierarchies with self-consistent sources

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Abstract

Using gauge transformations for the corresponding generating pseudo-differential operators L^n in terms of eigenfunctions and adjoint eigenfunctions, we construct several types of auto-Bäcklund transformations for the KP hierarchy with self-consistent sources (KPHSCS) and mKP hierarchy with self-consistent sources (mKPHSCS), respectively. The Bäcklund transformations from KPHSCS to mKPHSCS are also constructed in this way.

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1. Introduction

Soliton equations with self-consistent sources (SESCSs) are important models in many fields of physics, such as hydrodynamics, solid state physics, plasma physics, etc [5–13]. For example, the KP equation with self-consistent sources (KPESCS) describes the interaction of a long wave with a short-wave packet propagating on the x, y plane at an angle to each other (see [12] and the references therein). Until now, several methods have been developed in solving the soliton hierarchies with self-consistent sources (SHSCS) in (1+1) dimensions, such as the inverse scattering transform (IST), Darboux transformation (DT) and so on (see [5–7, 9–19]). Extension to (2+1) dimensions of the established methods investigated for SHSCSs in (1+1) dimensions is the subject of the current research. Recently, by treating the (2+1)-dimensional constrained soliton hierarchies [30, 35] as the stationary ones of the corresponding hierarchies with self-consistent sources, we develop a systematical way to find integrable (2+1)-dimensional soliton hierarchies with self-consistent sources and their Lax representations. For example, in [36] and [37], the integrable KP hierarchy with self-consistent sources (KPHSCS) and mKP hierarchy with self-consistent sources (mKPHSCS) together with their Lax representations are obtained, and the generalized binary Darboux transformations with arbitrary functions in time t for the KP equation with self-consistent sources (KPESCS) and mKP equation with self-consistent sources (mKEHSCS) are constructed to obtain some interesting solutions.

Bäcklund transformations generated via gauge invariance have been shown to be a powerful tool to investigate the soliton hierarchies in the last three decades. The (2+1)-dimensional Bäcklund transformations employed were constructed by the so-called dressing method based on appropriate gauge transformation [22]. This dressing approach had been earlier used to construct auto-Bäcklund transformations for the KP, two-dimensional three wave and Davey–Stewartson equations in turn [20, 21]. Konopelchenko, Oevel and co-workers extended this method to investigate some important (2+1)-dimensional integrable hierarchies constructed in the framework of Sato theory, such as the KP hierarchy, the mKP hierarchy and the Dym hierarchy [23–33]. Via gauge transformations utilizing eigenfunctions and adjoint eigenfunctions, auto-Bäcklund transformations and Bäcklund transformations between these hierarchies are constructed which reveal the intimate connections between them. Also, gauge transformations are applicable to investigate the constrained KP and constrained mKP hierarchies [27, 30, 32, 33]. These Bäcklund transformations map the bi-Hamiltonian structure of the constrained KP hierarchy to that of the constrained mKP hierarchy [30, 32, 33].

From the Darboux transformations for the KP equation [34] and mKP equation [38], the auto-Bäcklund transformations for the first equation in the KPHSCS and mKPHSCS, i.e., for the KPESCS and mKPESCS, have been constructed respectively in [36] and [37]. But the auto-Bäcklund transformations for these two hierarchies with self-consistent sources and the relation between them still remain unknown. In this paper, we are devoted to finding the gauge invariance of the KPHSCS and mKPHSCS and the relation between them. Since in our approach we regard the constrained hierarchy as the stationary hierarchy of the corresponding hierarchy with self-consistent sources, the known information about the constrained hierarchy may be suggestive for us to obtain the information about the corresponding hierarchy with self-consistent sources. The constrained KP hierarchy (cKPH) studied in [27, 32, 33] may be treated as the stationary hierarchy of the KPHSCS, so it is straightforward for us to generalize the auto-Bäcklund transformations for the cKPH studied in [27, 33] to those for the KPHSCS. Though the constrained mKP hierarchy (cmKPH) considered in [30, 32, 33] is different from the stationary hierarchy of the mKPHSCS in our case, the idea of constructing Bäcklund transformations for it is still helpful. In this paper, utilizing the associated eigenfunctions and adjoint eigenfunctions, we construct several types of auto-Bäcklund transformations for the KPHSCS and mKPHSCS, respectively. Their compositions and iteration are formulated. The Bäcklund transformations between them are also considered. As pointed out in the paper, the results obtained here will recover some known results for some degenerate cases obtained in [27, 32, 33].

The paper is organized as follows. First, we briefly review some notation and useful identities of the pseudo-differential operators (PDO) in section 2 and give some general relations for PDOs under various gauge transformations in section 3. In the framework of Sato theory, the KPHSCS and mKPHSCS and their conjugate Lax pairs are introduced in section 4. Based on Darboux covariance of the Lax pairs, we construct several types of auto-Bäcklund transformations for the KPHSCS and mKPHSCS, respectively via gauge transformations in terms of the associated eigenfunctions and adjoint eigenfunctions in sections 5 and 6. The Bäcklund transformations from the KPHSCS to mKPHSCS are also considered in section 7.

2. Some notation and identities about the PDO

We will discuss the KPHSCS and mKPHSCS in the framework of Sato theory. First, we will give some basic notation and identities about the PDO which will be used in our following discussions. More details about the PDO will be referred to [1–4].

For a PDO of the form

$$\Lambda = \sum_{i<\infty} u_i \partial^i,$$

where $\partial = \partial_x$, $u_i \in g_0$ and g_0 is a differential algebra, we have the following notation:

$$(\Lambda)_{\geqslant k} = \sum_{i\geqslant k} u_i \partial^i, \qquad (\Lambda)_{<k} = \sum_{i<k} u_i \partial^i, \qquad (\Lambda)_k = u_k \partial^k,$$

$$\operatorname{res}(\Lambda) = u_{-1}, \qquad (\Lambda)^* = \sum_{i<\infty} (-1)^i \partial^i u_i.$$

For a given function f, Λf denotes the composition of Λ with the multiplication operator f while $\Lambda(f)$ denotes the action of the differential part of Λ on f, i.e.,

$$\Lambda(f) = (\Lambda f)_0 = [(\Lambda)_{\geqslant 0} f]_0 = \sum_{i\geqslant 0} u_i f^{(i)}.$$

Some identities of the PDO are also useful and we list them below:

$$(\Lambda^*)_0 = \operatorname{res}(\partial^{-1}\Lambda), \qquad (\Lambda)_0 = \operatorname{res}(\Lambda\partial^{-1}), \qquad (\Lambda\partial^{-1})_{<0} = (\Lambda)_0\partial^{-1} + (\Lambda)_{<0}\partial^{-1},$$

$$[(\Lambda)_{\geqslant 0}]^* = [\Lambda^*]_{\geqslant 0}, \qquad (\partial^{-1}\Lambda)_{<0} = \partial^{-1}(\Lambda^*)_0 + \partial^{-1}(\Lambda)_{<0}.$$

3. Gauge transformations

Here we will give some results about gauge transformations for the PDOs from which the Bäcklund transformations for the KPHSCS and mKPHSCS originate.

Lemma 3.1. *For arbitrary PDO A, functions f and g, the following identities hold:*

(1) $(f^{-1}Af)_{\geqslant 1} = f^{-1}(A)_{\geqslant 0} f - f^{-1}(A)_{\geqslant 0}(f)$,

(2) $(f\partial f^{-1}Af\partial^{-1}f^{-1})_{\geqslant 0} = f\partial f^{-1}(A)_{\geqslant 0} f\partial^{-1}f^{-1} - f(f^{-1}(A)_{\geqslant 0}(f))_x\partial^{-1}f^{-1}$,

(3) $(f^{-1}Af)_{\geqslant 1} = f^{-1}(A)_{\geqslant 1} f - f^{-1}(A)_{\geqslant 1}(f)$,

(4) $\left(f_x^{-1}\partial A\partial^{-1}f_x\right)_{\geqslant 1} = f_x^{-1}\partial(A)_{\geqslant 1}\partial^{-1}f_x - f_x^{-1}((A)_{\geqslant 1}(f))_x$,

(5) $(\partial^{-1}gAg^{-1}\partial)_{\geqslant 1} = \partial^{-1}g(A)_{\geqslant 0}g^{-1}\partial - \partial^{-1}g^{-1}(A)^*_{\geqslant 0}(g)\partial$,

(6) $(g^{-1}\partial^{-1}gAg^{-1}\partial g)_{\geqslant 0} = g^{-1}\partial^{-1}g(A)_{\geqslant 0}g^{-1}\partial g + g^{-1}\partial^{-1}g[g^{-1}(A_{\geqslant 0})^*(g)]_x$,

(7) $\left(\partial^{-1}g_xAg_x^{-1}\partial\right)_{\geqslant 1} = \partial^{-1}g_x(A)_{\geqslant 1}g_x^{-1}\partial - \partial^{-1}g_x^{-1}[(A)_{\geqslant 1}]^*(g_x)\partial$,

(8) $(\partial^{-1}g\partial A\partial^{-1}g^{-1}\partial)_{\geqslant 1} = \partial^{-1}g\partial(A)_{\geqslant 1}\partial^{-1}g^{-1}\partial - \partial^{-1}g^{-1}[\partial(A)_{\geqslant 1}\partial^{-1}]^*(g)\partial$.

The results of (1)–(4) are given by Oevel and Rogers in [28] and the remaining results can be proved directly.

Lemma 3.2. *Let L be an arbitrary PDO and $f(x, t_q) \neq 0$, $g(x, t_q)$ be arbitrary functions; the following identities hold:*

(1) If $\tilde{L} = f^{-1}Lf$, $\tilde{f} = f^{-1}g$ and $\tilde{g} = D^{-1}(fg)$, where D^{-1} defines the integral operation $D^{-1}(f) = \int^x f(\xi)\,\mathrm{d}\xi$, then

$$(\tilde{L}^q)_{\geqslant 1} = [f^{-1}(L^q)_{\geqslant 0} f]_{\geqslant 1},$$

$$\tilde{f}_{t_q} - (\tilde{L}^q)_{\geqslant 1}(\tilde{f}) = -f^{-2}g\left[f_{t_q} - (L^q)_{\geqslant 0}(f)\right] + f^{-1}\left[g_{t_q} - (L^q)_{\geqslant 0}(g)\right],$$

$$\begin{aligned}\tilde{g}_{t_q} + [\partial(\tilde{L}^q)_{\geqslant 1}\partial^{-1}]^*(\tilde{g}) &= \tilde{g}_{t_q} + D^{-1}([(\tilde{L}^q)_{\geqslant 1}]^*(\tilde{g}_x)) \\ &= D^{-1}\left[f\left(g_{t_q} + (L^q)^*_{\geqslant 0}(g)\right)\right] + D^{-1}\left[g\left(f_{t_q} - (L^q)_{\geqslant 0}(f)\right)\right].\end{aligned}$$

(2) *If* $\tilde{L} = f\partial f^{-1}Lf\partial^{-1}f^{-1}$, $\tilde{f} = f(f^{-1}g)_x$ *and* $\tilde{g} = f^{-1}D^{-1}(fg)$, *then*

$$\begin{aligned}
&(\tilde{L}^q)_{\geqslant 0} = [f\partial f^{-1}(L^q)_{\geqslant 0}f\partial^{-1}f^{-1}]_{\geqslant 0},\\
&\tilde{f}_{t_q} - (\tilde{L}^q)_{\geqslant 0}(\tilde{f}) = -g\big[f^{-1}\big(f_{t_q} - (L^q)_{\geqslant 0}(f)\big)\big]_x + f\big[f^{-1}\big(g_{t_q} - (L^q)_{\geqslant 0}(g)\big)\big]_x,\\
&\tilde{g}_{t_q} + (\tilde{L}^q)^*_{\geqslant 0}(\tilde{g}) = -f^{-2}D^{-1}(fg)\big[f_{t_q} - (L^q)_{\geqslant 0}(f)\big] + f^{-1}D^{-1}\big[f\big(g_{t_q} + (L^q)^*_{\geqslant 0}(g)\big)\big]\\
&\qquad\qquad + f^{-1}D^{-1}\big[g\big(f_{t_q} - (L^q)_{\geqslant 0}(f)\big)\big].
\end{aligned}$$

(3) *If* $\tilde{L} = f^{-1}Lf$, $\tilde{f} = f^{-1}g$ *and* $\tilde{g} = D^{-1}(fg_x)$, *then*

$$\begin{aligned}
&(\tilde{L}^q)_{\geqslant 1} = [f^{-1}(L^q)_{\geqslant 1}f]_{\geqslant 1},\\
&\tilde{f}_{t_q} - (\tilde{L}^q)_{\geqslant 1}(\tilde{f}) = -f^{-2}g\big[f_{t_q} - (L^q)_{\geqslant 1}(f)\big] + f^{-1}\big[g_{t_q} - (L^q)_{\geqslant 1}(g)\big],\\
&\tilde{g}_{t_q} + [\partial(\tilde{L}^q)_{\geqslant 1}\partial^{-1}]^*(\tilde{g}) = D^{-1}\big[f\big(g_{t_q} + (\partial(L^q)_{\geqslant 1}\partial^{-1})^*(g)\big)\big]\\
&\qquad\qquad + D^{-1}\big[g_x\big(f_{t_q} - (L^q)_{\geqslant 1}(f)\big)\big].
\end{aligned}$$

(4) *If* $\tilde{L} = f_x^{-1}\partial L\partial^{-1}f_x$, $\tilde{f} = f_x^{-1}g_x$ *and* $\tilde{g} = D^{-1}(f_xg)$, *then*

$$\begin{aligned}
&(\tilde{L}^q)_{\geqslant 1} = \big[f_x^{-1}\partial(L^q)_{\geqslant 1}\partial^{-1}f_x\big]_{\geqslant 1},\\
&\tilde{f}_{t_q} - (\tilde{L}^q)_{\geqslant 1}(\tilde{f}) = -f_x^{-2}g_x\big[f_{t_q} - (L^q)_{\geqslant 1}(f)\big]_x + f_x^{-1}\big[g_{t_q} - (L^q)_{\geqslant 1}(g)\big]_x,\\
&\tilde{g}_{t_q} + [\partial(\tilde{L}^q)_{\geqslant 1}\partial^{-1}]^*(\tilde{g}) = D^{-1}\big[f_x\big(g_{t_q} + (\partial(L^q)_{\geqslant 1}\partial^{-1})^*(g)\big)\big]\\
&\qquad\qquad + D^{-1}\big[g\big(f_{t_q} - (L^q)_{\geqslant 1}(f)\big)_x\big].
\end{aligned}$$

Parts of the above results (for $\tilde{f}$) are given by Oevel and Rogers in [28]; the remaining results for $\tilde{g}$ can be proved by directly following lemma 3.1.

Lemma 3.3. *Let L be an arbitrary PDO and* $f(x, t_q), g(x, t_q) \neq 0$ *be arbitrary functions; the following identities hold:*

(1) *If* $\tilde{L} = \partial^{-1}gLg^{-1}\partial$, $\tilde{f} = D^{-1}(fg)$ *and* $\tilde{g} = fg^{-1}$, *then*

$$\begin{aligned}
&(\tilde{L}^q)_{\geqslant 1} = [\partial^{-1}g(L^q)_{\geqslant 0}g^{-1}\partial]_{\geqslant 1},\\
&\tilde{f}_{t_q} - (\tilde{L}^q)_{\geqslant 1}(\tilde{f}) = D^{-1}\big[g\big(f_{t_q} - (L^q)_{\geqslant 0}(f)\big)\big] + D^{-1}\big[f\big(g_{t_q} + (L^q)^*_{\geqslant 0}(g)\big)\big],\\
&\tilde{g}_{t_q} + [\partial(\tilde{L}^q)_{\geqslant 1}\partial^{-1}]^*(\tilde{g}) = g^{-1}\big[f_{t_q} + (L^q)^*_{\geqslant 0}(f)\big] - g^{-2}f\big[g_{t_q} + (L^q)^*_{\geqslant 0}(g)\big].
\end{aligned}$$

(2) *If* $\tilde{L} = g^{-1}\partial^{-1}gLg^{-1}\partial g$, $\tilde{f} = g^{-1}D^{-1}(fg)$ *and* $\tilde{g} = g(g^{-1}f)_x$, *then*

$$\begin{aligned}
&(\tilde{L}^q)_{\geqslant 0} = [g^{-1}\partial^{-1}g(L^q)_{\geqslant 0}g^{-1}\partial g]_{\geqslant 0},\\
&\tilde{f}_{t_q} - (\tilde{L}^q)_{\geqslant 0}(\tilde{f}) = -g^{-2}D^{-1}(fg)\big[g_{t_q} + (L^q)^*_{\geqslant 0}(g)\big] + g^{-1}D^{-1}\big[g\big(f_{t_q} - (L^q)_{\geqslant 0}(f)\big)\big]\\
&\qquad\qquad + g^{-1}D^{-1}\big[f\big(g_{t_q} + (L^q)^*_{\geqslant 0}(g)\big)\big],\\
&\tilde{g}_{t_q} + (\tilde{L}^q)^*_{\geqslant 0}(\tilde{g}) = g\big[g^{-1}\big(f_{t_q} + (L^q)^*_{\geqslant 0}(f)\big)\big]_x - f\big[g^{-1}\big(g_{t_q} + (L^q)^*_{\geqslant 0}(g)\big)\big]_x.
\end{aligned}$$

(3) *If* $\tilde{L} = \partial^{-1}g_xLg_x^{-1}\partial$, $\tilde{f} = D^{-1}(fg_x)$ *and* $\tilde{g} = f_xg_x^{-1}$, *then*

$$\begin{aligned}
&(\tilde{L}^q)_{\geqslant 1} = \big[\partial^{-1}g_x(L^q)_{\geqslant 1}g_x^{-1}\partial\big]_{\geqslant 1},\\
&\tilde{f}_{t_q} - (\tilde{L}^q)_{\geqslant 1}(\tilde{f}) = D^{-1}\big[f\big(g_{t_q} + (\partial(L^q)_{\geqslant 1}\partial^{-1})^*(g)\big)_x\big] + D^{-1}\big[g_x\big(f_{t_q} - (L^q)_{\geqslant 1}(f)\big)\big],\\
&\tilde{g}_{t_q} + [\partial(\tilde{L}^q)_{\geqslant 1}\partial^{-1}]^*(\tilde{g}) = g_x^{-1}\big[f_{t_q} + (\partial(L^q)_{\geqslant 1}\partial^{-1})^*(f)\big]_x\\
&\qquad\qquad - g_x^{-2}f_x\big[g_{t_q} + (\partial(L^q)_{\geqslant 1}\partial^{-1})^*(g)\big]_x.
\end{aligned}$$

(4) If $\tilde{L} = \partial^{-1} g \partial L \partial^{-1} g^{-1} \partial$, $\tilde{f} = D^{-1}(f_x g)$ *and* $\tilde{g} = f g^{-1}$, *then*

$$(\tilde{L}^q)_{\geqslant 1} = [\partial^{-1} g \partial (L^q)_{\geqslant 1} \partial^{-1} g^{-1} \partial]_{\geqslant 1},$$

$$\tilde{f}_{t_q} - (\tilde{L}^q)_{\geqslant 1}(\tilde{f}) = D^{-1}\big[f_x\big(g_{t_q} + (\partial (L^q)_{\geqslant 1} \partial^{-1})^*(g)\big)\big] + D^{-1}\big[g\big(f_{t_q} - (L^q)_{\geqslant 1}(f)\big)_x\big],$$

$$\tilde{g}_{t_q} + [\partial (\tilde{L}^q)_{\geqslant 1} \partial^{-1}]^*(\tilde{g}) = g^{-1}\big[f_{t_q} + (\partial (L^q)_{\geqslant 1} \partial^{-1})^*(f)\big] - g^{-2} f\big[g_{t_q} + (\partial (L^q)_{\geqslant 1} \partial^{-1})^*(g)\big].$$

These results can be proved by directly following lemma 3.1.

4. The KPHSCS and mKPHSCS

(1) Consider the PDO of the form

$$L = L_{KP} = \partial + (u(t)/2)\partial^{-1} + u_1(t)\partial^{-2} + \cdots \qquad t = (t_1 = x, t_2, \ldots), \tag{4.1}$$

which satisfies the constraint

$$L^n = (L^n)_{\geqslant 0} + \sum_{i=1}^{N} q_i(t)\partial^{-1} r_i(t) = \partial^n + \omega_{n-2}(t)\partial^{n-2} + \cdots + \omega_0(t) + \sum_{i=1}^{N} q_i(t)\partial^{-1} r_i(t),$$
$$\omega_{n-2} = \frac{n}{2}u, \tag{4.2}$$

with $q_i(t), r_i(t)$ satisfying

$$(q_i)_{t_k} = (L^k)_{\geqslant 0}(q_i), \qquad (r_i)_{t_k} = -(L^k)^*_{\geqslant 0}(r_i), \qquad k, n \in \mathbb{N},$$

where

$$(L^k)_{\geqslant 0} = [(L^n)^{\frac{k}{n}}]_{\geqslant 0} = \left[\left((L^n)_{\geqslant 0} + \sum_{i=1}^{N} q_i \partial^{-1} r_i\right)^{\frac{k}{n}}\right]_{\geqslant 0}.$$

Starting from the PDO L^n (4.2) and requiring $k < n$, the KPHSCS is defined by the following Lax representation [36]:

$$((L^k)_{\geqslant 0})_{t_n} - (L^n)_{t_k} + [(L^k)_{\geqslant 0}, L^n] = 0, \tag{4.3a}$$

$$(q_i)_{t_k} = (L^k)_{\geqslant 0}(q_i), \tag{4.3b}$$

$$(r_i)_{t_k} = -(L^k)^*_{\geqslant 0}(r_i), \qquad i = 1, \ldots, N. \tag{4.3c}$$

Under (4.3b) and (4.3c), (4.3a) will be obtained by the compatibility condition of either

$$\psi_{t_k} = (L^k)_{\geqslant 0}(\psi), \tag{4.4a}$$

$$\psi_{t_n} = (L^n)_{\geqslant 0}(\psi) + \sum_{i=1}^{N} q_i D^{-1}(r_i \psi), \tag{4.4b}$$

or

$$\bar{\psi}_{t_k} = -(L^k)^*_{\geqslant 0}(\bar{\psi}), \tag{4.5a}$$

$$\bar{\psi}_{t_n} = -(L^n)^*_{\geqslant 0}(\bar{\psi}) + \sum_{i=1}^{N} r_i D^{-1}(q_i \bar{\psi}). \tag{4.5b}$$

ψ and $\bar{\psi}$ will be called the eigenfunction and adjoint eigenfunction of the KPHSCS (4.3), respectively. Equations (4.3) will be often written as equations of the fields u, q_i and r_i when the auxiliary fields $\omega_{n-3}, \ldots, \omega_0$ are eliminated. For example, when $k = 2, n = 3$, we will get the KPESCS as [12, 36]

$$\left[4u_t - 6uu_x - u_{xxx} + 8\left(\sum_{j=1}^{N} q_j r_j\right)_x\right]_x - 3u_{yy} = 0, \tag{4.6a}$$

$$q_{j,y} = q_{j,xx} + uq_j, \tag{4.6b}$$

$$r_{j,y} = -r_{j,xx} - ur_j, \qquad j = 1, \ldots, N. \tag{4.6c}$$

(2) Consider another PDO of the form

$$L = L_{mKP} = \partial + v(t) + v_1(t)\partial^{-1} + \cdots, \tag{4.7}$$

which satisfies the constraint

$$L^n = (L^n)_{\geqslant 1} + \sum_{i=1}^{N} q_i(t)\partial^{-1} r_i(t)\partial = \partial^n + \pi_{n-1}(t)\partial^{n-1} + \cdots + \pi_1(t)\partial + \sum_{i=1}^{N} q_i(t)\partial^{-1} r_i(t)\partial,$$

$$\pi_{n-1} = nv, \tag{4.8}$$

with $q_i(t), r_i(t)$ satisfy

$$(q_i)_{t_k} = (L^k)_{\geqslant 1}(q_i), \qquad (r_i)_{t_k} = -(\partial(L^k)_{\geqslant 1}\partial^{-1})^*(r_i), \qquad k, n \in \mathbb{N},$$

where

$$(L^k)_{\geqslant 1} = [(L^n)^{\frac{k}{n}}]_{\geqslant 1} = \left[\left((L^n)_{\geqslant 1} + \sum_{i=1}^{N} q_i \partial^{-1} r_i \partial\right)^{\frac{k}{n}}\right]_{\geqslant 1}.$$

Starting from the PDO (4.8) and requiring $k < n$, the mKPHSCS is defined by the following Lax representation [37]:

$$((L^k)_{\geqslant 1})_{t_n} - (L^n)_{t_k} + [(L^k)_{\geqslant 1}, L^n] = 0, \tag{4.9a}$$

$$(q_i)_{t_k} = (L^k)_{\geqslant 1}(q_i), \tag{4.9b}$$

$$(r_i)_{t_k} = -(\partial(L^k)_{\geqslant 1}\partial^{-1})^*(r_i), \qquad i = 1, \ldots, N. \tag{4.9c}$$

Under (4.9*b*) and (4.9*c*), (4.9*a*) will be obtained by the compatibility condition of either

$$\phi_{t_k} = (L^k)_{\geqslant 1}(\phi), \tag{4.10a}$$

$$\phi_{t_n} = (L^n)_{\geqslant 1}(\phi) + \sum_{i=1}^{N} q_i D^{-1}(r_i \phi_x), \tag{4.10b}$$

or

$$\bar{\phi}_{t_k} = -(\partial(L^k)_{\geqslant 1}\partial^{-1})^*(\bar{\phi}), \tag{4.11a}$$

$$\bar{\phi}_{t_n} = -(\partial(L^n)_{\geqslant 1}\partial^{-1})^*(\bar{\phi}) - \sum_{i=1}^{N} r_i D^{-1}(q_i \bar{\phi}_x). \tag{4.11b}$$

ϕ and $\bar{\phi}$ will be called the eigenfunction and adjoint eigenfunction of the mKPHSCS (4.9), respectively. Equations (4.9) will be often written as equations of the fields v, q_i and r_i when the auxiliary fields $\pi_{n-2}, \ldots, \pi_1$ are eliminated. For example, when $k = 2, n = 3$, we will get the mKPESCS as [37]

$$4v_t - v_{xxx} - 3D^{-1}(v_{yy}) - 6D^{-1}(v_y)v_x + 6v^2v_x + 4\sum_{i=1}^{N}(q_ir_i)_x = 0, \qquad (4.12a)$$

$$q_{i,y} = q_{i,xx} + 2vq_{i,x}, \qquad (4.12b)$$

$$r_{i,y} = -r_{i,xx} + 2vr_{i,x}, \qquad i = 1, \ldots, N. \qquad (4.12c)$$

The identities shown in section 3 lead in a natural way to invariances and relations of the KPHSCS and mKPHSCS. We will discuss them in the following sections.

5. The auto-Bäcklund transformations for the KPHSCS

1. Auto-Bäcklund transformation utilizing the eigenfunction.

Theorem 5.1. *Let L^n of (4.2) satisfy the KPHSCS (4.3) and ψ be the corresponding eigenfunction. The function $f \neq 0$ satisfies (4.4).*

Define

$$T_1[f] : L^n \overset{f}{\longmapsto} \tilde{L}^n$$

by

$$\tilde{L}^n = \left(\chi_1 L^n \chi_1^{-1}\right)_{\geqslant 0} + \sum_{i=1}^{N} \tilde{q}_i \partial^{-1}\tilde{r}_i, \qquad (5.1a)$$

where

$$\chi_1 = f\partial f^{-1}, \qquad \tilde{q}_i = f(f^{-1}q_i)_x, \qquad \tilde{r}_i = -f^{-1}D^{-1}(r_if), \qquad i = 1, \ldots, N. \qquad (5.1b)$$

Then $\tilde{L}^n$ will also satisfy the KPHSCS (4.3). So, $T_1[f]$ defines an auto-Bäcklund transformation for the KPHSCS (4.3).

Proof. It is straightforward to see $(\tilde{L}^n)_{\geqslant 0}$ has the same form as $(L^n)_{\geqslant 0}$, i.e.,

$$(\tilde{L}^n)_{\geqslant 0} = \left(\chi_1 L^n \chi_1^{-1}\right)_{\geqslant 0} = \partial^n + \tilde{\omega}_{n-2}\partial^{n-2} + \cdots + \tilde{\omega}_0, \qquad \tilde{\omega}_{n+2} = \frac{n\tilde{u}}{2} = \frac{n}{2}(u + 2(\ln f)_{xx}).$$

Furthermore, we can find that for $k < n$,

$$(\tilde{L}^k)_{\geqslant 0} = [(\tilde{L}^n)^{\frac{k}{n}}]_{\geqslant 0} = [((\tilde{L}^n)_{\geqslant 0})^{\frac{k}{n}}]_{\geqslant 0} = \left[\left(\left(\chi_1 L^n \chi_1^{-1}\right)_{\geqslant 0}\right)^{\frac{k}{n}}\right]_{\geqslant 0} = \left[\left(\chi_1 L^n \chi_1^{-1}\right)^{\frac{k}{n}}\right]_{\geqslant 0}$$
$$= \left[\chi_1 (L^n)^{\frac{k}{n}} \chi_1^{-1}\right]_{\geqslant 0} = \left[\chi_1 (L^k)_{\geqslant 0} \chi_1^{-1}\right]_{\geqslant 0}.$$

Define $\tilde{\psi} = f(f^{-1}\psi)_x$. From lemma 3.2 (2), we can easily see that $\tilde{q}_i, \tilde{r}_i$ and $\tilde{\psi}$ satisfy (4.3*b*), (4.3*c*) and (4.4*a*) w.r.t. $\tilde{L}^n$, respectively. Besides,

$$\tilde{\psi}_{t_n} - (\tilde{L}^n)_{\geqslant 0}(\tilde{\psi}) = -\psi\left[f^{-1}\left(f_{t_n} - (L^n)_{\geqslant 0}(f)\right)\right]_x + f\left[f^{-1}\left(\psi_{t_n} - (L^n)_{\geqslant 0}(\psi)\right)\right]_x$$
$$= -\psi\left[f^{-1}\sum_{i=1}^{N} q_i D^{-1}(r_if)\right]_x + f\left[f^{-1}\sum_{i=1}^{N} q_i D^{-1}(r_i\psi)\right]_x$$
$$= -\psi\sum_{i=1}^{N}(f^{-1}q_i)_x D^{-1}(r_if) + f\sum_{i=1}^{N}(f^{-1}q_i)_x D^{-1}(r_i\psi)$$
$$= -\sum_{i=1}^{N}(f^{-1}q_i)_x f[f^{-1}\psi D^{-1}(r_if) - D^{-1}(r_iff^{-1}\psi)]$$

$$= \sum_{i=1}^{N} f(f^{-1}q_i)_x D^{-1}[-f^{-1}D^{-1}(r_i f) f (f^{-1}\psi)_x]$$
$$= \sum_{i=1}^{N} \tilde{q}_i D^{-1}(\tilde{r}_i \tilde{\psi}),$$

so $\tilde{\psi}$ satisfies (4.4*b*) w.r.t. $\tilde{L}^n$.

This completes the proof. □

(2) Auto-Bäcklund transformation utilizing the adjoint eigenfunction.

Theorem 5.2. *Let L^n of (4.2) satisfy the KPHSCS (4.3) and ψ be the corresponding eigenfunction. The function $g \neq 0$ satisfies (4.5).*

Define

$$T_2[g] : L^n \stackrel{g}{\longmapsto} \tilde{L}^n$$

by

$$\tilde{L}^n = \left(\chi_2 L^n \chi_2^{-1}\right)_{\geqslant 0} + \sum_{i=1}^{N} \tilde{q}_i \partial^{-1} \tilde{r}_i, \tag{5.2a}$$

where

$$\chi_2 = g^{-1}\partial^{-1} g, \qquad \tilde{q}_i = g^{-1}D^{-1}(g q_i), \qquad \tilde{r}_i = -g(r_i g^{-1})_x, \qquad i = 1, \ldots, N. \tag{5.2b}$$

Then $\tilde{L}^n$ will also satisfy the KPHSCS (4.3). So, $T_2[g]$ defines an auto-Bäcklund transformation for the KPHSCS (4.3).

Proof. It is straightforward to see $(\tilde{L}^n)_{\geqslant 0}$ has the same form as $(L^n)_{\geqslant 0}$, i.e.,

$$(\tilde{L}^n)_{\geqslant 0} = \left(\chi_2 L^n \chi_2^{-1}\right)_{\geqslant 0} = \partial^n + \tilde{\omega}_{n-2}\partial^{n-2} + \cdots + \tilde{\omega}_0, \qquad \tilde{\omega}_{n+2} = \frac{n\tilde{u}}{2} = \frac{n}{2}(u + 2(\ln g)_{xx}).$$

Furthermore, we can find that for $k < n$,

$$(\tilde{L}^k)_{\geqslant 0} = [(\tilde{L}^n)^{\frac{k}{n}}]_{\geqslant 0} = [((\tilde{L}^n)_{\geqslant 0})^{\frac{k}{n}}]_{\geqslant 0} = \left[\left(\left(\chi_2 L^n \chi_2^{-1}\right)_{\geqslant 0}\right)^{\frac{k}{n}}\right]_{\geqslant 0} = \left[\left(\chi_2 L^n \chi_2^{-1}\right)^{\frac{k}{n}}\right]_{\geqslant 0}$$
$$= \left[\chi_2 (L^n)^{\frac{k}{n}} \chi_2^{-1}\right]_{\geqslant 0} = \left[\chi_2 (L^k)_{\geqslant 0} \chi_2^{-1}\right]_{\geqslant 0}.$$

Define $\tilde{\psi} = g^{-1}D^{-1}(g\psi)$. From lemma 3.3 (2), we can easily see that $\tilde{q}_i, \tilde{r}_i$ and $\tilde{\psi}$ satisfy (4.3*b*), (4.3*c*) and (4.4*a*) w.r.t. $\tilde{L}^n$, respectively. Besides,

$$\tilde{\psi}_{t_n} - (\tilde{L}^n)_{\geqslant 0}(\tilde{\psi}) = -g^{-2}D^{-1}(g\psi)\left[g_{t_n} + (L^n)^*_{\geqslant 0}(g)\right] + g^{-1}D^{-1}\left[g\left(\psi_{t_n} - (L^n)_{\geqslant 0}(\psi)\right)\right]$$
$$+\, g^{-1}D^{-1}\left[\psi\left(g_{t_n} + (L^n)^*_{\geqslant 0}(g)\right)\right]$$
$$= -g^{-2}D^{-1}(g\psi)\sum_{i=1}^{N} r_i D^{-1}(g q_i)$$
$$+\, g^{-1}D^{-1}\left[g \sum_{i=1}^{N} q_i D^{-1}(r_i\psi)\right] + g^{-1}D^{-1}\left[\psi \sum_{i=1}^{N} r_i D^{-1}(g q_i)\right]$$
$$= -g^{-1}\sum_{i=1}^{N} D^{-1}(g q_i) D^{-1}(r_i\psi) - g^{-2}\sum_{i=1}^{N} D^{-1}(g\psi) D^{-1}(g q_i) r_i$$

$$= -\sum_{i=1}^{N} g^{-1}D^{-1}(gq_i)[-D^{-1}(r_i g^{-1} g\psi) + g^{-1} r_i D^{-1}(g\psi)]$$

$$= -\sum_{i=1}^{N} g^{-1}D^{-1}(gq_i)D^{-1}[(g^{-1}r_i)_x D^{-1}(g\psi)]$$

$$= \sum_{i=1}^{N} \tilde{q}_i D^{-1}(\tilde{r}_i\tilde{\psi}),$$

so $\tilde{\psi}$ satisfies (4.4*b*) w.r.t. $\tilde{L}^n$.

This completes the proof. □

(3) T: composition of T_1 and T_2.

Let L^n of (4.2) be a solution of the KPHSCS (4.3), $f \neq 0$ and $g \neq 0$ satisfy (4.4) and (4.5), respectively. Utilizing f, $T_1[f]$ transforms L^n into $\tilde{L}^n = (\tilde{L}^n)_{\geqslant 0} + \sum_{i=1}^{N} \tilde{q}_i \partial^{-1} \tilde{r}_i$ which is a new solution of the KPHSCS (4.3). It is easy to prove that $\tilde{g} = f^{-1}D^{-1}(fg)$ satisfies (4.5) w.r.t. $\tilde{L}^n$. So utilizing $\tilde{g}$ again, $\tilde{L}^n$ will be transformed by $T_2[\tilde{g}]$ into $\hat{L}^n = (\hat{L}^n)_{\geqslant 0} + \sum_{j=1}^{N} \hat{q}_j \partial^{-1} \hat{r}_j = \partial^n + \hat{\omega}_{n-2}\partial^{n-2} + \cdots + \hat{\omega}_0 + \sum_{j=1}^{N} \hat{q}_j \partial^{-1} \hat{r}_j$ $\left(\hat{\omega}_{n-2} = \frac{n}{2}\hat{u}\right)$ which will satisfy the KPHSCS (4.3) again. We list the results below.

$$\hat{L}^n \triangleq T[f,g](L^n) = (T_2[\tilde{g}] \circ T_1[f])(L^n) = (\chi L^n \chi^{-1})_{\geqslant 0} + \sum_{j=1}^{N} \hat{q}_j \partial^{-1} \hat{r}_j, \quad \chi = \tilde{g}^{-1}\partial^{-1}\tilde{g} f \partial f^{-1} \qquad (\tilde{g} = f^{-1}D^{-1}(fg)), \tag{5.3a}$$

$$\hat{u} = u + 2[\ln(D^{-1}(fg))]_{xx}, \tag{5.3b}$$

$$\hat{q}_i \triangleq T[f,g](q_i) = (T_2[\tilde{g}] \circ T_1[f])(q_i) = q_i - \frac{fD^{-1}(q_i g)}{D^{-1}(fg)}, \tag{5.3c}$$

$$\hat{r}_i \triangleq T[f,g](q_i) = (T_2[\tilde{g}] \circ T_1[f])(r_i) = r_i - \frac{gD^{-1}(r_i f)}{D^{-1}(fg)}. \tag{5.3d}$$

Remark.

1. When $((L^k)_{\geqslant 0})_{t_n} = 0$, (4.3) will degenerate to the constrained KP hierarchy. Correspondingly, $T_1[f]$ with f satisfying (4.4*a*) and $T_2[g]$ with g satisfying (4.5*a*) will give auto-Bäcklund transformations for the constrained KP hierarchy (strictly, not the whole hierarchy because $k < n$). Different from our case in that the auto-Bäcklund transformations here are constructed directly for L^n (4.2), those constructed in [27] are for L (4.1). So in order to preserve the constraint form of (4.2), f used in [27] has to satisfy an extra eigenvalue problem $(L^n)_{\geqslant 0}(f) + \sum_{i=1}^{N} q_i D^{-1}(r_i f) = \lambda f$.
2. When $q_j = r_j = 0$, $j = 1, \ldots, N$, (4.3) will degenerate to the KP hierarchy. $T_1[f]$, $T_2[g]$ and $T[f,g]$ introduced above will degenerate to the auto-Bäcklund transformations for the KP hierarchy based on gauge transformations which have been widely studied in [24–31](f and g now only need to be the eigenfunction and adjoint eigenfunction of the KP hierarchy, i.e., they satisfy (4.4) and (4.5), respectively, with $q_i = r_i = 0$).
3. When $k = 2$ and $n = 3$, $T_1[f]$, $T_2[g]$ and $T[f,g]$ introduced above will give rise to the auto-Bäcklund transformations for the KPESCS (4.6) induced by the backward, forward and binary Darboux transformations, respectively [36] (f and g now only need to be the eigenfunction and adjoint eigenfunction of the KPESCS, respectively).

4. *Iteration of T*. Assuming $\psi_1, \ldots, \psi_n$ are n arbitrary solutions of (4.4) and $\bar{\psi}_1, \ldots, \bar{\psi}_n$ are n arbitrary solutions of (4.5), we define the following Wronskians (Gramm determinants):

$$\begin{aligned} &W_1(\psi_1, \ldots, \psi_n; \bar{\psi}_1, \ldots, \bar{\psi}_n) = \det(U_{n\times n}), \\ &W_2(\psi_1, \ldots, \psi_n; \bar{\psi}_1, \ldots, \bar{\psi}_{n-1}) = \det(V_{n\times n}), \\ &W_3(\psi_1, \ldots, \psi_{n-1}; \bar{\psi}_1, \ldots, \bar{\psi}_n) = \det(X_{n\times n}), \end{aligned} \tag{5.4}$$

where

$$U_{i,j} = D^{-1}(\bar{\psi}_i\psi_j), \qquad i, j = 1, \ldots, n, \tag{5.5a}$$

$$\begin{aligned} &V_{i,j} = U_{i,j}, && i = 1, \ldots, n-1, \quad j = 1, \ldots, n, \\ &V_{n,j} = \psi_j, && j = 1, \ldots, n, \end{aligned} \tag{5.5b}$$

$$\begin{aligned} &X_{i,j} = D^{-1}(\bar{\psi}_j\psi_i), && i = 1, \ldots, n-1, \quad j = 1, \ldots, n, \\ &X_{n,j} = \bar{\psi}_j, && j = 1, \ldots, n. \end{aligned} \tag{5.5c}$$

We denote $u[n]$, $q_i[n]$ and $r_i[n]$ as the n times iteration of T on u, q_i and r_i, respectively; then the following formulae hold.

$$u[n] = u + 2\partial^2 \ln W_1(f_1, \ldots, f_n; g_1, \ldots, g_n), \tag{5.6a}$$

$$q_i[n] = \frac{W_2(f_1, \ldots, f_n, q_i; g_1, \ldots, g_n)}{W_1(f_1, \ldots, f_n; g_1, \ldots, g_n)}, \tag{5.6b}$$

$$r_i[n] = \frac{W_3(f_1, \ldots, f_n; g_1, \ldots, g_n, r_i)}{W_1(f_1, \ldots, f_n; g_1, \ldots, g_n)}, \tag{5.6c}$$

where f_i and g_i, $i = 1, \ldots, n$ are arbitrary solutions of (4.4) and (4.5), respectively.

We can prove (5.6) in the same way as we did in [36] and we omit it here.

6. The auto-Bäcklund transformations for the mKPHSCS

1. Auto-Bäcklund transformations utilizing the eigenfunction.

Theorem 6.1. *Let L^n of (4.8) satisfy the mKPHSCS (4.9) and ϕ is the corresponding eigenfunction. The function $f \neq 0$ satisfies (4.10). Define*

$$G_1[f] : L^n \overset{f}{\longmapsto} \tilde{L}^n$$

by

$$\tilde{L}^n = (f^{-1}L^n f)_{\geqslant 1} + \sum_{j=1}^{N} \tilde{q}_j\partial^{-1}\tilde{r}_j\partial, \qquad \tilde{q}_j = f^{-1}q_j, \qquad \tilde{r}_j = D^{-1}(r_{j,x}f),$$
$$j = 1, \ldots, N. \tag{6.1}$$

Then $\tilde{L}^n$ will also satisfy the mKPHSCS (4.9).

Proof. It is straightforward to see $(\tilde{L}^n)_{\geqslant 1}$ has the same form as $(L^n)_{\geqslant 1}$, i.e.,

$$(\tilde{L}^n)_{\geqslant 1} = (f^{-1}L^n f)_{\geqslant 1} = \partial^n + \tilde{\pi}_{n-1}\partial^{n-1} + \cdots + \tilde{\pi}_1\partial, \qquad \tilde{\pi}_{n-1} = n\tilde{v} = n(v + (\ln f)_x).$$

Furthermore we can find that for $k < n$,

$$\begin{aligned} (\tilde{L}^k)_{\geqslant 1} &= [(\tilde{L}^n)^{\frac{k}{n}}]_{\geqslant 1} = [((\tilde{L}^n)_{\geqslant 1})^{\frac{k}{n}}]_{\geqslant 1} = [((f^{-1}L^n f)_{\geqslant 1})^{\frac{k}{n}}]_{\geqslant 1} \\ &= [(f^{-1}L^n f)^{\frac{k}{n}}]_{\geqslant 1} = [f^{-1}(L^n)^{\frac{k}{n}} f]_{\geqslant 1} = [f^{-1}(L^k)_{\geqslant 1} f]_{\geqslant 1}. \end{aligned}$$

Define $\tilde{\phi} = f^{-1}\phi$. From lemma 3.2 (3), we can easily see that $\tilde{q}_i, \tilde{r}_i, i = 1, \ldots, N$, and $\tilde{\phi}$ satisfy (4.9b), (4.9c) and (4.10a) w.r.t. $\tilde{L}^n$, respectively. Besides,

$$\begin{aligned}\tilde{\phi}_{t_n} - (\tilde{L}^n)_{\geqslant 1}(\tilde{\phi}) &= -f^{-2}\phi\big[f_{t_n} - (L^n)_{\geqslant 1}(f)\big] + f^{-1}\big[\phi_{t_n} - (L^n)_{\geqslant 1}(\phi)\big]\\ &= -f^{-2}\phi\sum_{i=1}^{N} q_i D^{-1}(r_i f_x) + f^{-1}\left[\sum_{i=1}^{N} q_i D^{-1}(r_i\phi_x)\right]\\ &= f^{-2}\phi\sum_{i=1}^{N} q_i D^{-1}(r_{i,x} f) - f^{-1}\sum_{i=1}^{N} q_i D^{-1}(r_{i,x}\phi)\\ &= \sum_{i=1}^{N}\tilde{q}_i D^{-1}(\tilde{r}_i\tilde{\phi}_x).\end{aligned}$$

So $\tilde{\phi}$ satisfies (4.10b) w.r.t. $\tilde{L}^n$. This completes the proof. □

Theorem 4. *Let L^n of (4.8) satisfy the mKPHSCS (4.9) and ϕ be the corresponding eigenfunction. The function $f \neq 0$ satisfies (4.10). Define*

$$G_2[f] : L^n \overset{f}{\longmapsto} \tilde{L}^n$$

by

$$\tilde{L}^n = \left(f_x^{-1}\partial L^n \partial^{-1} f_x\right)_{\geqslant 1} + \sum_{j=1}^{N}\tilde{q}_j\partial^{-1}\tilde{r}_j\partial, \qquad \tilde{q}_j = f_x^{-1}q_{j,x}, \qquad \tilde{r}_j = -D^{-1}(r_j f_x),$$
$$j = 1, \ldots, N. \tag{6.2}$$

Then $\tilde{L}^n$ will also satisfy the mKPHSCS (4.9).

Proof. It is straightforward to see $(\tilde{L}^n)_{\geqslant 1}$ has the same form as $(L^n)_{\geqslant 1}$, i.e.,

$$(\tilde{L}^n)_{\geqslant 1} = \left(f_x^{-1}\partial L^n\partial^{-1} f_x\right)_{\geqslant 1} = \partial^n + \tilde{\pi}_{n-1}\partial^{n-1} + \cdots + \tilde{\pi}_1\partial,$$
$$\tilde{\pi}_{n-1} = n\tilde{v} = n(v + (\ln f_x)_x).$$

Furthermore we can find that for $k < n$,

$$\begin{aligned}(\tilde{L}^k)_{\geqslant 1} &= [(\tilde{L}^n)^{\frac{k}{n}}]_{\geqslant 1} = [((\tilde{L}^n)_{\geqslant 1})^{\frac{k}{n}}]_{\geqslant 1} = \left[\left(\left(f_x^{-1}\partial L^n\partial^{-1} f_x\right)_{\geqslant 1}\right)^{\frac{k}{n}}\right]_{\geqslant 1}\\ &= \left[\left(f_x^{-1}\partial L^n\partial^{-1} f_x\right)^{\frac{k}{n}}\right]_{\geqslant 1} = \left[f_x^{-1}\partial(L^n)^{\frac{k}{n}}\partial^{-1} f_x\right]_{\geqslant 1} = \left[f_x^{-1}\partial(L^k)_{\geqslant 1}\partial^{-1} f_x\right]_{\geqslant 1}.\end{aligned}$$

Define $\tilde{\phi} = f_x^{-1}\phi_x$. From lemma 3.2 (4), we can easily see that $\tilde{q}_i, \tilde{r}_i, i = 1, \ldots, N$, and $\tilde{\phi}$ satisfy (4.9b), (4.9c) and (4.10a) w.r.t. $\tilde{L}^n$, respectively. Besides,

$$\begin{aligned}\tilde{\phi}_{t_n} - (\tilde{L}^n)_{\geqslant 1}(\tilde{\phi}) &= -f_x^{-2}\phi_x\big[f_{t_n} - (L^n)_{\geqslant 1}(f)\big]_x + f_x^{-1}\big[\phi_{t_n} - (L^n)_{\geqslant 1}(\phi)\big]_x\\ &= -f_x^{-2}\phi_x\left[\sum_{i=1}^{N} q_i D^{-1}(r_i f_x)\right]_x + f_x^{-1}\left[\sum_{i=1}^{N} q_i D^{-1}(r_i\phi_x)\right]_x\\ &= -f_x^{-2}\phi_x\sum_{i=1}^{N} q_{i,x} D^{-1}(r_i f_x) + f_x^{-1}\sum_{i=1}^{N} q_{i,x} D^{-1}(r_i\phi_x)\\ &= \sum_{i=1}^{N}\tilde{q}_i D^{-1}(\tilde{r}_i\tilde{\phi}_x),\end{aligned}$$

so $\tilde{\phi}$ satisfies (4.10b) w.r.t. $\tilde{L}^n$.

This completes the proof. □

2. Auto-Bäcklund transformations utilizing the adjoint eigenfunction.

Theorem 6.3. *Let L^n of (4.8) satisfy the mKPHSCS (4.9) and ϕ be the corresponding eigenfunction. The function g ($g_x \neq 0$) satisfies (4.11). Define*

$$G_3[g] : L^n \stackrel{g}{\longmapsto} \tilde{L}^n$$

by

$$\tilde{L}^n = \left(\partial^{-1} g_x L^n g_x^{-1} \partial\right)_{\geqslant 1} + \sum_{j=1}^{N} \tilde{q}_j \partial^{-1} \tilde{r}_j \partial, \qquad \tilde{q}_j = -D^{-1}(g_x q_j), \qquad \tilde{r}_j = r_{j,x} g_x^{-1},$$
$$j = 1, \ldots, N. \tag{6.3}$$

Then $\tilde{L}^n$ will also satisfy the mKPHSCS (4.9).

Proof. It is straightforward to see $(\tilde{L}^n)_{\geqslant 1}$ has the same form as $(L^n)_{\geqslant 1}$, i.e.,

$$(\tilde{L}^n)_{\geqslant 1} = \left(\partial^{-1} g_x L^n g_x^{-1} \partial\right)_{\geqslant 1} = \partial^n + \tilde{\pi}_{n-1} \partial^{n-1} + \cdots + \tilde{\pi}_1 \partial,$$
$$\tilde{\pi}_{n-1} = n\tilde{v} = n(v - (\ln g_x)_x).$$

Furthermore we can find that for $k < n$,

$$(\tilde{L}^k)_{\geqslant 1} = [(\tilde{L}^n)^{\frac{k}{n}}]_{\geqslant 1} = [((\tilde{L}^n)_{\geqslant 1})^{\frac{k}{n}}]_{\geqslant 1} = \left[\left(\left(\partial^{-1} g_x L^n g_x^{-1} \partial\right)_{\geqslant 1}\right)^{\frac{k}{n}}\right]_{\geqslant 1} = \left[\left(\partial^{-1} g_x L^n g_x^{-1} \partial\right)^{\frac{k}{n}}\right]_{\geqslant 1}$$
$$= \left[\partial^{-1} g_x (L^n)^{\frac{k}{n}} g_x^{-1} \partial\right]_{\geqslant 1} = \left[\partial^{-1} g_x (L^k)_{\geqslant 1} g_x^{-1} \partial\right]_{\geqslant 1}.$$

Define $\tilde{\phi} = D^{-1}(g_x \phi)$. From lemma 3.3 (3), we can easily see that $\tilde{q}_i, \tilde{r}_i, i = 1, \ldots, N$, and $\tilde{\phi}$ satisfy (4.9*b*), (4.9*c*) and (4.10*a*) w.r.t. $\tilde{L}^n$, respectively. Besides,

$$\begin{aligned}
&\tilde{\phi}_{t_n} - (\tilde{L}^n)_{\geqslant 1}(\tilde{\phi}) \\
&\quad = D^{-1}\left[g_x\left(\phi_{t_n} - (L^n)_{\geqslant 1}(\phi)\right)\right] + D^{-1}\left[\phi\left(g_{t_n} + (\partial (L^n)_{\geqslant 1} \partial^{-1})^*(g)\right)_x\right] \\
&\quad = D^{-1}\left[g_x\left(\sum_{i=1}^{N} q_i D^{-1}(r_i \phi_x)\right)\right] + D^{-1}\left[\phi\left(-\sum_{i=1}^{N} r_i D^{-1}(q_i g_x)\right)_x\right] \\
&\quad = D^{-1}\left[\sum_{i=1}^{N} g_x q_i (r_i \phi - D^{-1}(r_{i,x}\phi))\right] - D^{-1}\left[\sum_{i=1}^{N} \phi r_{i,x} D^{-1}(q_i g_x) + \phi r_i g_x q_i\right] \\
&\quad = -\sum_{i=1}^{N} D^{-1}[D^{-1}(r_{i,x}\phi) q_i g_x + \phi r_{i,x} D^{-1}(q_i g_x)] \\
&\quad = -\sum_{i=1}^{N} D^{-1}(r_{i,x}\phi) D^{-1}(q_i g_x) \\
&\quad = \sum_{i=1}^{N} \tilde{q}_i D^{-1}(\tilde{r}_i \tilde{\phi}_x),
\end{aligned}$$

so $\tilde{\phi}$ satisfies (4.10*b*) w.r.t. $\tilde{L}^n$.

This completes the proof. □

Theorem 6.4. *Let L^n of (4.8) satisfy the mKPHSCS (4.9) and ϕ be the corresponding eigenfunction. The function $g \neq 0$ satisfies (4.11). Define*

$$G_4[g] : L^n \stackrel{g}{\longmapsto} \tilde{L}^n$$

by

$$\tilde{L}^n = (\partial^{-1} g \partial L^n \partial^{-1} g^{-1} \partial)_{\geqslant 1} + \sum_{j=1}^{N} \tilde{q}_j \partial^{-1} \tilde{r}_j \partial, \qquad \tilde{q}_j = D^{-1}(g q_{j,x}), \qquad \tilde{r}_j = r_j g^{-1},$$
$$j = 1, \ldots, N. \tag{6.4}$$

Then $\tilde{L}^n$ will also satisfy the mKPHSCS (4.9).

Proof. It is straightforward to see $(\tilde{L}^n)_{\geqslant 1}$ has the same form as $(L^n)_{\geqslant 1}$, i.e.,

$$(\tilde{L}^n)_{\geqslant 1} = (\partial^{-1} g \partial L^n \partial^{-1} g^{-1} \partial)_{\geqslant 1} = \partial^n + \tilde{\pi}_{n-1} \partial^{n-1} + \cdots + \tilde{\pi}_1 \partial,$$
$$\tilde{\pi}_{n-1} = n\tilde{v} = n(v - (\ln g)_x).$$

Furthermore we can find that for $k < n$,

$$\begin{aligned}(\tilde{L}^k)_{\geqslant 1} &= [(\tilde{L}^n)^{\frac{k}{n}}]_{\geqslant 1} = [((\tilde{L}^n)_{\geqslant 1})^{\frac{k}{n}}]_{\geqslant 1} = [((\partial^{-1} g \partial L^n \partial^{-1} g^{-1} \partial)_{\geqslant 1})^{\frac{k}{n}}]_{\geqslant 1} \\ &= [(\partial^{-1} g \partial L^n \partial^{-1} g^{-1} \partial)^{\frac{k}{n}}]_{\geqslant 1} = [\partial^{-1} g \partial (L^n)^{\frac{k}{n}} \partial^{-1} g^{-1} \partial]_{\geqslant 1} \\ &= [\partial^{-1} g \partial (L^k)_{\geqslant 1} \partial^{-1} g^{-1} \partial]_{\geqslant 1}.\end{aligned}$$

Define $\tilde{\phi} = D^{-1}(g\phi_x)$. From lemma 3.3 (4), we can easily see that $\tilde{q}_i, \tilde{r}_i, i = 1, \ldots, N$, and $\tilde{\phi}$ satisfy (4.9*b*), (4.9*c*) and (4.10*a*) w.r.t. $\tilde{L}^n$, respectively. Besides,

$$\begin{aligned}\tilde{\phi}_{t_n} - (\tilde{L}^n)_{\geqslant 1}(\tilde{\phi}) &= D^{-1}\big[g\big(\phi_{t_n} - (L^n)_{\geqslant 1}(\phi)\big)_x\big] + D^{-1}\big[\phi_x\big(g_{t_n} + (\partial (L^n)_{\geqslant 1} \partial^{-1})^*(g)\big)\big] \\ &= D^{-1}\left[g\left(\sum_{i=1}^{N} q_i D^{-1}(r_i \phi_x)\right)_x\right] + D^{-1}\left[\phi_x\left(-\sum_{i=1}^{N} r_i D^{-1}(q_i g_x)\right)\right] \\ &= D^{-1}\left[\sum_{i=1}^{N} g(q_i r_i \phi_x + q_{i,x} D^{-1}(r_i \phi_x))\right] \\ &\quad - D^{-1}\left[\sum_{i=1}^{N} \phi_x r_i (g q_i - D^{-1}(q_{i,x} g))\right] \\ &= \sum_{i=1}^{N} D^{-1}(g q_{i,x}) D^{-1}(\phi_x r_i) \\ &= \sum_{i=1}^{n} \tilde{q}_i D^{-1}(\tilde{r}_i \tilde{\phi}_x),\end{aligned}$$

so $\tilde{\phi}$ satisfies (4.10*b*) w.r.t. $\tilde{L}^n$.

This completes the proof. □

3. Compositions of G_i.

(1) G_{12}: composition of G_1 and G_2.

Let L^n of (4.8) satisfy the mKPHSCS (4.9), f_1 ($f_1 \neq 0$ and $f_{1,x} \neq 0$) and $f_2 = 1$ satisfy (4.10). Utilizing f_1, $G_1[f_1]$ transforms L^n into $\tilde{L}^n = (\tilde{L}^n)_{\geqslant 1} + \sum_{j=1}^{N} \tilde{q}_j \partial^{-1} \tilde{r}_j \partial$, a new solution of the mKPHSCS (4.9). It is not difficult to prove that $\tilde{f}_2 = f_1^{-1} f_2 = f_1^{-1}$ satisfies (4.10) w.r.t. $\tilde{L}^n$. So utilizing $\tilde{f}_2$ again, $G_2[\tilde{f}_2]$ transforms $\tilde{L}^n$ into another solution $\hat{L}^n = (\hat{L}^n)_{\geqslant 1} + \sum_{j=1}^{N} \hat{q}_j \partial^{-1} \hat{r}_j \partial$. We list the results below.

$$\begin{aligned}\hat{L}^n &\triangleq G_{12}[f_1](L^n) = (G_2[\tilde{f}_2] \circ G_1[f_1])(L^n) \\ &= \left(f_{1,x}^{-1} f_1^2 \partial f_1^{-1} L^n f_1 \partial^{-1} f_1^{-2} f_{1,x}\right)_{\geqslant 1} + \sum_{j=1}^{N} \hat{q}_j \partial^{-1} \hat{r}_j \partial,\end{aligned} \tag{6.5a}$$

$$\hat{q}_j \triangleq G_{12}[f_1](q_j) = (G_2[\tilde{f}_2] \circ G_1[f_1])(q_j) = f_{1,x}^{-1}(f_{1,x}q_j - f_1 q_{j,x}), \tag{6.5b}$$

$$\hat{r}_j \triangleq G_{12}[f_1](r_j) = (G_2[\tilde{f}_2] \circ G_1[f_1])(r_j) = f_1^{-1} D^{-1}(r_j f_{1,x}), \quad j = 1, \ldots, N. \tag{6.5c}$$

(2) G_{34}: composition of G_3 and G_4.
Let L^n of (4.8) satisfy the mKPHSCS (4.9), g_1 ($g_1 \neq 0$ and $g_{1,x} \neq 0$) and $g_2 = 1$ satisfy (4.11). Utilizing g_1, $G_3[g_1]$ transforms L^n into $\tilde{L}^n = (\tilde{L}^n)_{\geqslant 1} + \sum_{j=1}^N \tilde{q}_j \partial^{-1} \tilde{r}_j \partial$, a new solution of the mKPHSCS (4.9). It is not difficult to prove that $\tilde{g}_2 = g_1^{-1} g_2 = g_1^{-1}$ satisfies (4.11) w.r.t. $\tilde{L}^n$. So utilizing $\tilde{g}_2$ again, $G_4[\tilde{g}_2]$ transforms $\tilde{L}^n$ into $\hat{L}^n = (\hat{L}^n)_{\geqslant 1} + \sum_{j=1}^N \hat{q}_j \partial^{-1} \hat{r}_j \partial$. We list the results below.

$$\hat{L}^n \triangleq G_{34}[g_1](L^n) = (G_4[\tilde{g}_2] \circ G_3[g_1])(L^n)$$
$$= \left(\partial^{-1} g_{1,x} g_1^{-2} \partial^{-1} g_1 \partial L^n \partial^{-1} g_1^{-1} \partial g_{1,x}^{-1} g_1^2 \partial\right)_{\geqslant 1} + \sum_{j=1}^N \tilde{q}_j \partial^{-1} \tilde{r}_j \partial, \tag{6.6a}$$

$$\hat{q}_j \triangleq G_{34}[g_1](q_j) = (G_4[\tilde{g}_2] \circ G_3[g_1])(q_j) = g_1^{-1} D^{-1}(q_j g_{1,x}), \tag{6.6b}$$

$$\hat{r}_j \triangleq G_{34}[g_1](q_j) = (G_4[\tilde{g}_2] \circ G_3[g_1])(r_j) = g_{1,x}^{-1}(g_{1,x} r_j - g_1 r_{j,x}),$$
$$j = 1, \ldots, N. \tag{6.6c}$$

(3) G: composition of G_{12} and G_{34}.
Let L^n of (4.8) satisfy the mKPHSCS (4.9), f ($f \neq 0$ and $f_x \neq 0$) and g ($g \neq 0$ and $g_x \neq 0$) satisfy (4.10) and (4.11) respectively. Utilizing f, $G_{12}[f]$ transforms L^n into $\tilde{L}^n = (\tilde{L}^n)_{\geqslant 1} + \sum_{j=1}^N \tilde{q}_j \partial^{-1} \tilde{r}_j \partial$. It is not difficult to verify that $\tilde{g} = G_{12}[f](g) = f^{-1} D^{-1}(g f_x)$ satisfies (4.11) w.r.t. $\tilde{L}^n$. So utilizing $\tilde{g}$ again, $G_{34}[\tilde{g}]$ transforms $\tilde{L}^n$ into another solution of mKPHSCS (4.9) $\hat{L}^n$. We list the results below.

$$\hat{L}^n \triangleq G[f,g](L^n) = (G_{34}[\tilde{g}] \circ G_{12}[f])(L^n) = (\sigma L^n \sigma^{-1})_{\geqslant 1} + \sum_{j=1}^N \hat{q}_j \partial^{-1} \hat{r}_j \partial$$
$$= \partial^n + \hat{\pi}_{n-1} \partial^{n-1} + \cdots + \hat{\pi}_1 \partial + \sum_{j=1}^N \hat{q}_j \partial^{-1} \hat{r}_j \partial, \tag{6.7a}$$

$$\sigma = \partial^{-1} \tilde{g}_x \tilde{g}^{-2} \partial^{-1} \tilde{g} \partial f_x^{-1} f^2 \partial f^{-1} \qquad (\tilde{g} = f^{-1} D^{-1}(g f_x)),$$

$$\hat{\pi}_{n-1} = n\hat{v}, \qquad \hat{v} = v - \left[\ln\left(\frac{D^{-1}(f g_x)}{D^{-1}(f_x g)}\right)\right]_x, \tag{6.7b}$$

$$\hat{q}_j \triangleq G[f,g](q_j) = (G_{34}[\tilde{g}] \circ G_{12}[f])(q_j) = q_j - \frac{f D^{-1}(g q_{j,x})}{D^{-1}(g f_x)}, \tag{6.7c}$$

$$\hat{r}_j \triangleq G[f,g](r_j) = (G_{34}[\tilde{g}] \circ G_{12}[f])(r_j) = r_j - \frac{g D^{-1}(f r_{j,x})}{D^{-1}(f g_x)}, \qquad j = 1, \ldots, N. \tag{6.7d}$$

Remark.

1. When $q_j = r_j = 0$, $j = 1, \ldots, N$, G_i, G_{kl} and G introduced above will degenerate to the auto-Bäcklund transformations for the mKP hierarchy based on gauge transformations which have been widely studied in [24–31]. Now f only need to be an eigenfunction of the mKP hierarchy, i.e., it satisfies (4.10) with $q_i = r_i = 0$. g only need to be an adjoint eigenfunction (in some articles, often called integrated adjoint eigenfuction) of the mKP hierarchy, i.e., satisfies (4.11) with $q_i = r_i = 0$.
2. When $k = 2$ and $N = 3$, $G[f, g]$ introduced above will give rise to the auto-Bäcklund transformation for the mKPESCS (4.12) induced by binary Darboux transformation introduced in [38] (f and g now only need to be the eigenfunction and adjoint eigenfunction of the mKPESCS, respectively).
3. *Iteration of G.* Assuming $\phi_1, \ldots, \phi_n$ are n arbitrary solutions of (4.10) and $\bar{\phi}_1, \ldots, \bar{\phi}_n$ are n arbitrary solutions of (4.11), we define the following Wronskians (Gramm determinants):

$$\begin{aligned} &W_1(\phi_1, \ldots, \phi_n; \bar{\phi}_1, \ldots, \bar{\phi}_n) = \det(Y_{n\times n}),\\ &W_2(\phi_1, \ldots, \phi_n; \bar{\phi}_1, \ldots, \bar{\phi}_n) = \det(\tilde{Y}_{n\times n}),\\ &W_3(\phi_1, \ldots, \phi_n; \bar{\phi}_1, \ldots, \bar{\phi}_{n-1}) = \det(Z_{n\times n}),\\ &W_4(\phi_1, \ldots, \phi_{n-1}; \bar{\phi}_1, \ldots, \bar{\phi}_n) = \det(\tilde{Z}_{n\times n}), \end{aligned} \tag{6.8}$$

where

$$Y_{i,j} = D^{-1}(\bar{\phi}_j\phi_{i,x}), \qquad i, j = 1, \ldots, n, \tag{6.9a}$$

$$\tilde{Y}_{i,j} = D^{-1}(\bar{\phi}_{j,x}\phi_i), \qquad i, j = 1, \ldots, n, \tag{6.9b}$$

$$\begin{aligned} &Z_{i,j} = D^{-1}(\bar{\phi}_i\phi_{j,x}), \qquad i = 1, \ldots, n-1, \quad j = 1, \ldots, n,\\ &Z_{n,j} = \phi_j, \qquad j = 1, \ldots, n, \end{aligned} \tag{6.9c}$$

$$\begin{aligned} &\tilde{Z}_{i,j} = D^{-1}(\bar{\phi}_{j,x}\phi_i), \qquad i = 1, \ldots, n-1, \quad j = 1, \ldots, n,\\ &\tilde{Z}_{n,j} = \bar{\phi}_j, \qquad j = 1, \ldots, n. \end{aligned} \tag{6.9d}$$

We denote $v[n]$, $q_i[n]$ and $r_i[n]$ as the n times iteration of G on v, q_i and r_i, respectively; then the following formulae hold.

$$v[n] = v - \partial_x \ln \frac{W_2(f_1, \ldots, f_n; g_1, \ldots, g_n)}{W_1(f_1, \ldots, f_n; g_1, \ldots, g_n)}, \tag{6.10a}$$

$$q_i[n] = \frac{W_3(f_1, \ldots, f_n, q_i; g_1, \ldots, g_n)}{W_1(f_1, \ldots, f_n; g_1, \ldots, g_n)}, \tag{6.10b}$$

$$r_i[n] = \frac{W_4(f_1, \ldots, f_n; g_1, \ldots, g_n, r_i)}{W_2(f_1, \ldots, f_n; g_1, \ldots, g_n)}, \tag{6.10c}$$

where f_i and g_i, $i = 1, \ldots, n$ are arbitrary solutions of (4.10) and (4.11), respectively. We can prove (6.10) in the same way as we did in [37] and we omit it here.

7. The Bäcklund transformations between the KPHSCS and mKPHSCS

(1) Bäcklund transformation utilizing eigenfunction of the KPHSCS.

Theorem 7.1. *Let L^n of (4.2) satisfy the KPHSCS (4.3) and ψ be the corresponding eigenfunction. The function $f \neq 0$ satisfies (4.4). Define*

$$M_1[f] : L^n \overset{f}{\longmapsto} \tilde{L}^n$$

by

$$\tilde{L}^n = (f^{-1}L^n f)_{\geqslant 1} + \sum_{j=1}^{N} \tilde{q}_j \partial^{-1} \tilde{r}_j \partial, \qquad \tilde{q}_j = f^{-1} q_j, \qquad \tilde{r}_j = -D^{-1}(r_j f),$$
$$j = 1, \ldots, N. \tag{7.1}$$

Then $\tilde{L}^n$ will satisfy the mKPHSCS (4.9).

Proof. It is straightforward to see that

$$(\tilde{L}^n)_{\geqslant 1} = (f^{-1}L^n f)_{\geqslant 1} = \partial^n + \tilde{\pi}_{n-1}\partial^{n-1} + \cdots + \tilde{\pi}_1 \partial, \qquad \tilde{\pi}_{n-1} = n\tilde{v} = n((\ln f)_x).$$

Furthermore we can find that for $k < n$,

$$(\tilde{L}^k)_{\geqslant 1} = \left[(\tilde{L}^n)^{\frac{k}{n}}\right]_{\geqslant 1} = \left[((\tilde{L}^n)_{\geqslant 1})^{\frac{k}{n}}\right]_{\geqslant 1} = \left[((f^{-1}L^n f)_{\geqslant 1})^{\frac{k}{n}}\right]_{\geqslant 1} = \left[(f^{-1}L^n f)^{\frac{k}{n}}\right]_{\geqslant 1}$$
$$= \left[f^{-1}(L^n)^{\frac{k}{n}} f\right]_{\geqslant 1} = [f^{-1}(L^k)_{\geqslant 0} f]_{\geqslant 1}.$$

Define $\tilde{\phi} = f^{-1}\psi$. From lemma 3.2 (1), we can easily see that $\tilde{q}_i, \tilde{r}_i, i = 1, \ldots, N$, and $\tilde{\phi}$ satisfy (4.9*b*), (4.9*c*) and (4.10*a*) w.r.t. $\tilde{L}^n$, respectively. Besides,

$$\begin{aligned}
\tilde{\phi}_{t_n} - (\tilde{L}^n)_{\geqslant 1}(\tilde{\phi}) &= -f^{-2}\psi[f_{t_n} - (L^n)_{\geqslant 0}(f)] + f^{-1}[\psi_{t_n} - (L^n)_{\geqslant 0}(\psi)] \\
&= -f^{-2}\psi \sum_{i=1}^{N} q_i D^{-1}(r_i f) + f^{-1}\left[\sum_{i=1}^{N} q_i D^{-1}(r_i \psi)\right] \\
&= -\sum_{i=1}^{N} f^{-1} q_i [\tilde{\phi} D^{-1}(r_i f) - D^{-1}(r_i f \tilde{\phi})] \\
&= \sum_{i=1}^{N} \tilde{q}_i D^{-1}(\tilde{r}_i \tilde{\phi}_x),
\end{aligned}$$

so $\tilde{\phi}$ satisfies (4.10*b*) w.r.t. $\tilde{L}^n$.

This completes the proof. □

(2) Bäcklund transformation utilizing the adjoint eigenfunction of the KPHSCS.

Theorem 8. *Let L^n of (4.2) satisfy the KPHSCS (4.3) and ψ be the corresponding eigenfunction. The function $g \neq 0$ satisfies (4.5). Define*

$$M_2[g] : L^n \overset{g}{\longmapsto} \tilde{L}^n$$

by

$$\tilde{L}^n = (\partial^{-1} g L^n g^{-1} \partial)_{\geqslant 1} + \sum_{j=1}^{N} \tilde{q}_j \partial^{-1} \tilde{r}_j \partial, \qquad \tilde{q}_j = D^{-1}(g q_j), \qquad \tilde{r}_j = r_j g^{-1},$$
$$j = 1, \ldots, N. \tag{7.2}$$

Then $\tilde{L}^n$ will satisfy the mKPHSCS (4.9).

Proof. It is straightforward to see that

$$(\tilde{L}^n)_{\geqslant 1} = (\partial^{-1} g L^n g^{-1} \partial)_{\geqslant 1} = \partial^n + \tilde{\pi}_{n-1}\partial^{n-1} + \cdots + \tilde{\pi}_1 \partial, \qquad \tilde{\pi}_{n-1} = n\tilde{v} = -n((\ln g)_x).$$

Furthermore we can find that for $k < n$,

$$(\tilde{L}^k)_{\geqslant 1} = \left[(\tilde{L}^n)^{\frac{k}{n}}\right]_{\geqslant 1} = \left[((\tilde{L}^n)_{\geqslant 1})^{\frac{k}{n}}\right]_{\geqslant 1} = \left[((\partial^{-1}gL^ng^{-1}\partial)_{\geqslant 1})^{\frac{k}{n}}\right]_{\geqslant 1} = \left[(\partial^{-1}gL^ng^{-1}\partial)^{\frac{k}{n}}\right]_{\geqslant 1}$$
$$= \left[\partial^{-1}g(L^n)^{\frac{k}{n}}g^{-1}\partial\right]_{\geqslant 1} = [\partial^{-1}g(L^k)_{\geqslant 0}g^{-1}\partial]_{\geqslant 1}.$$

Define $\tilde{\phi} = D^{-1}(g\psi)$. From lemma 3.3 (1), we can easily see that $\tilde{q}_i, \tilde{r}_i, i = 1, \ldots, N$, and $\tilde{\phi}$ satisfy (4.9*b*), (4.9*c*) and (4.10*a*) w.r.t. $\tilde{L}^n$, respectively. Besides,

$$\begin{aligned}\tilde{\phi}_{t_n} - (\tilde{L}^n)_{\geqslant 1}(\tilde{\phi}) &= D^{-1}\left[\psi(g_{t_n} + (L^n)^*_{\geqslant 0}(g))\right] + D^{-1}\left[g\left(\psi_{t_n} - (L^n)_{\geqslant 0}(\psi)\right)\right] \\ &= D^{-1}\left[\psi\sum_{i=1}^N r_i D^{-1}(q_i g)\right] + D^{-1}\left[g\sum_{i=1}^N q_i D^{-1}(r_i\psi)\right] \\ &= \sum_{i=1}^N D^{-1}(gq_i)D^{-1}(r_i\psi) \\ &= \sum_{i=1}^N \tilde{q}_i D^{-1}(\tilde{r}_i\tilde{\phi}_x).\end{aligned}$$

so $\tilde{\phi}$ satisfies (4.10*b*) w.r.t. $\tilde{L}^n$.

This completes the proof. □

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